Geometric Series and Motorcycle Racing

by

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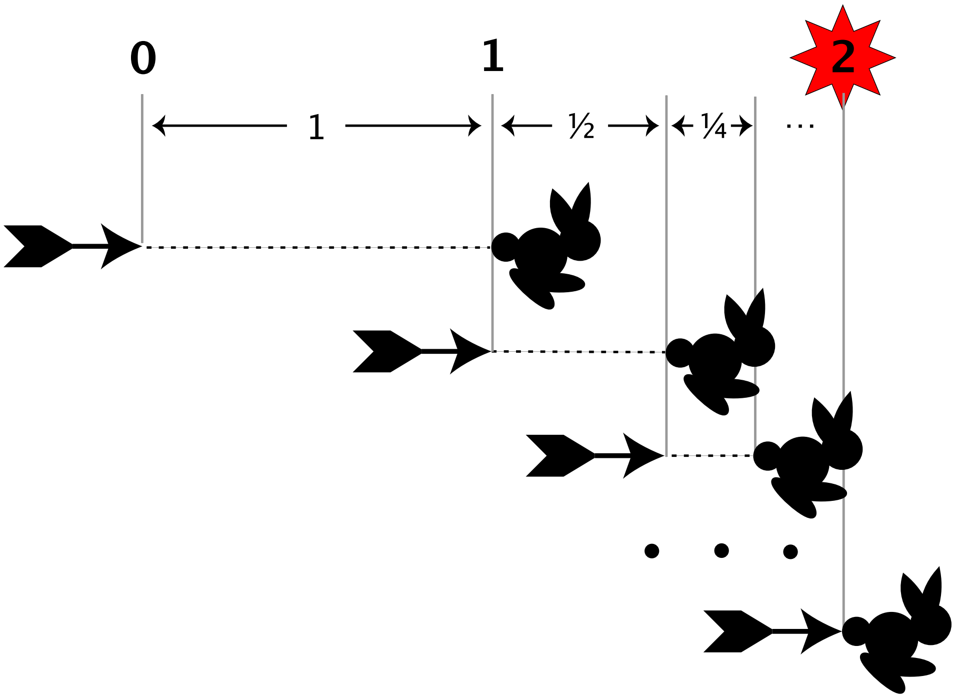
Zeno's Paradox (Achilles and the tortoise). You cannot shoot a rabbit running away from you—because whenever the arrow gets to where the rabbit was, the rabbit has moved further on.



Zeno's argument is a paradox. It, of course, isn't true. For example, if the arrow travels at twice the speed of the rabbit, then the arrow and the rabbit arrive at the same point (too bad for the rabbit) after the arrow has travelled to where the rabbit first was and then that distance once again.

Geometric Series. Consider the above arrow-rabbit problem. Let A denote the intial flight of the arrow, the distance the arrow travels to where the rabbit first was. The rabbit must have travelled $A/2$ further on when the arrow gets there since the rabbit moves at half the speed of the arrow. When the arrow traverses this distance, the rabbit is $(A/2)/2$ further on, and when the arrow covers this distance, the rabbit is still $((A/2)/2)/2$ ahead. And so it goes.

The arrow's progress is described by the sum of the successive distances it covers while chasing the rabbit:

$$A + A/2 + (A/2)/2 + ((A/2)/2)/2 + \cdots .$$

This infinite sum can be more conveniently written as

$$A + Ar + Ar^2 + Ar^3 + \cdots = A(1 + r + r^2 + r^3 + \cdots),$$

where $r = \frac{1}{2}$ is the ratio of successive summands. If and when the arrow covers all these distances, the rabbit will no longer be ahead; the arrow and the rabbit will have arrived at the same place at the same time (unfortunately for the rabbit).

Geometric series such as the above always have a finite sum when the (magnitude of the) ratio r is less than 1—that is to say when the arrow flies faster than the rabbit. To see this try multiplying the cubic partial sum $1+r+r^2+r^3$ by $1-r$. The $\pm r$, $\pm r^2$, $\pm r^3$ terms cancel each other out so that the product is $1 - r^4$. This kind of cancellation always happens when such a partial sum is multiplied by $1 - r$. Consequently the formula

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

is valid for each $n = 3, 4, \ldots$ But we are interested in the infinite sum, the sum when n becomes infinitely large. In this respect, when $|r| < 1$, the r^{n+1} on the right-hand side becomes smaller and smaller as n increases—until it disappears entirely in the infinite sum:

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}. \tag{1}$$

Now let's get back to the arrow-rabbit problem with the initial distance A. The arrow must travel

$$A + Ar + Ar^2 + Ar^3 + \cdots = \frac{A}{1 - r} \tag{2}$$

before it hits the rabbit. This amounts to a total distance of $2A$, after $r = \frac{1}{2}$ is plugged in.

Application to Motorcycle Racing. To compete in a top-level professional motorcyle race you have to qualify. Not just anyone is allowed to race. To compete in an American Superbike race you must be able to complete a qualifying lap within 108% of the fastest rider's time.

OK, suppose you have qualified at exactly 108% and the race is 28 laps long. You don't expect to win the race, but you certainly don't want to be lapped by anyone.

Let's look at the lapping situation. The fastest rider starts at the front of the pack. He rides at 108% of your speed. After he goes 1 lap you have only completed 1/1.08 of a lap. When he covers this 1/1.08 of a lap, you are $(1/1.08)^2$ laps ahead of him. And so it goes.

This is the arrow-rabbit problem all over again with $r = 1/1.08$; but the answer looks better in terms of the original percentage $p = 108\% = 1/r$. Under the 108% qualifying rule the leader laps you every

$$L = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r} = \frac{p}{p-1} = \frac{108\%}{8\%} = 13\frac{1}{2} \text{ laps.}$$

So much for not being lapped. You will be lapped at least once during the race, probably twice— if you don't have any problems.

The current 108% qualifying rule is a lot better than the 112% rule that used to apply to American Superbike racing. Not only did quite a few more riders qualify to race back then, but the borderline riders would be lapped every

$$L = \frac{p}{p-1} = \frac{112\%}{12\%} = 9\frac{1}{3} \text{ laps.}$$

Before 10 laps had been completed the leaders would not only be racing each other but simultaneously trying to work their way through a course of moving obstacles.

Algebra. One can also use high-school algebra to determine the relationship, $L = p/(p-1)$, between the qualifying percentage p and how often, in leader's laps L, a marginal qualifier is lapped. For this purpose let v_L and v_M denotes the speeds (in laps per hour) of the leader and the marginal qualifier, respectively. Then $v_L = pv_M$, and the time required for the marginal qualifier to be lapped (from the last time he was lapped) is

$$t = \frac{L}{v_L} = \frac{L-1}{v_M} \text{ hours.}$$

Multiplying both sides of the left-hand equation by v_L, substituting p for v_L/v_M, and solving for L in terms of p, we arrive at the formula $L = p/(p-1)$.

Your Preference. How you would solve the qualifying percentage problem—by geometric series or algebra—is up to you. I personally prefer the geometric series approach. It seems natural to the problem. Adding an infinite number of smaller and smaller pieces bothered Zeno over 2400 years ago. Hopefully we have made progress since then. Summing a geometric series shouldn't bother us now.

Besides, geometric series are the basis for the decimal representation of fractions. Most people realize that

$$\frac{1}{3} = 0.333333333333333333\ldots$$

ad infinitum. Fewer would recognize that

$$\frac{1}{7} = 0.142857142857142857\ldots.$$

In fact all fractions have such infinitely repeating blocks of digits in their decimal expansions, and the repeating part—unless it is an infinite string of zeros—corresponds to a geometric series with ratio a power of 1/10. Conversely, any infinite decimal expansion whose tail end is an infinitely repeating block of digits represents a rational number (a fraction).

For example take the decimal expansion of 1/7 above. Write out what what the right hand side means. Then apply formula (1) or (2) to the resulting geometric series to see that its sum is 1/7:

$$\begin{aligned}
&0.142857142857142857\ldots\\
&= \frac{142857}{1000000} + \frac{142857}{1000000^2} + \frac{142857}{1000000^3} + \ldots\\
&= \frac{142857}{1000000}\left\{1 + \frac{1}{1000000} + (\frac{1}{1000000})^2 + \ldots\right\}\\
&= \frac{142857}{1000000} \times \frac{1}{1 - \dfrac{1}{1000000}} \qquad \text{(by formula (1))}\\
&= \frac{142857}{999999} = \frac{1}{7}\,.
\end{aligned}$$

My point is this: geometric series, like fractions, are a part of the fabric of modern life—something that every young boy and girl should know. On the other hand, I have seen *college* students murder simple fractional arithmetic:

$$\frac{1}{3} + \frac{1}{7} = \frac{1+1}{3+7} = \frac{2}{10}\,.$$

Their mastery of geometric series may be simply too much to ask.