Some problems with the Markowitz mean-variance model

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ABSTRACT

Harry Markowitz's mean-variance model for portfolio choice posits a linear relationship between the return of a portfolio and the returns of its component securities. This linear relationship does not hold in an ex post setting when monthly or quarterly returns are used.

1 The Standard Portfolio Selection Model

Harry Markowitz begins *Mean-Variance Analysis in Portfolio Choice and Capital Markets* (Markowitz [1987]) with a description of the Standard Mean-Variance Portfolio Selection Model:

> an investor is to choose fractions $p_1, p_2, \ldots, p_n$ invested in n securities subject to constraints
>
> $$\sum_{j=1}^{n} p_j = 1, \qquad p_j \geq 0 \quad (j = 1, \ldots, n). \tag{1.1}$$
>
> We suppose that the returns this period on individual securities $r_1, r_2, \ldots, r_n$ are jointly distributed random variables, and the return on the portfolio is
>
> $$r = \sum_{j=1}^{n} r_j p_j . \tag{1.2}$$
>
> The expected (mean) return of the portfolio as a whole is
>
> $$\bar{r} = \sum_{j=1}^{n} \bar{r}_j p_j , \tag{1.3}$$
>
> where
>
> $$\bar{r}_j = E(r_j) \quad (j = 1, \ldots, n) \tag{1.4}$$
>
> and the E is the expected value operator.
>
> The variance of return on the portfolio is
>
> $$v = \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij} p_i p_j , \tag{1.5}$$
>
> where
>
> $$v_{ij} = E[(r_i - \bar{r}_i)(r_j - \bar{r}_j)] \tag{1.6}$$
>
> is the covariance between r_i and r_j.

This is *almost* an exact quote. The equation numbers are from Markowitz's book, but we have changed some of the symbols to suit our purposes.

2 Linearity

The standard model is a linear model. This means that the previous equations can be conveniently expressed in matrix form.

A portfolio is an n-vector (column matrix) $\mathbf{p} = [p_i](i = 1, \ldots, n)$ that satisfies

$$\sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0 \ (i = 1, \ldots, n). \tag{1}$$

If $R = [r_1, \ldots, r_n]$ denotes the row matrix of returns, then the return of the portfolio $\mathbf{p}$ is

$$r = R\,\mathbf{p}. \tag{2}$$

Here, of course, r and R are random variables, whereas $\mathbf{p}$ is a matrix of numbers.

The expected return of the portfolio $\mathbf{p}$ is

$$\bar{r} = E\,\mathbf{p}. \tag{3}$$

where

$$E = [\bar{r}_1, \ldots, \bar{r}_n] \tag{4}$$

is the row matrix of expected returns of the individual securities.

Finally, the variance of the portfolio can be written as

$$v = \mathbf{p}^T V \mathbf{p}, \tag{5}$$

where

$$V = [v_{ij}] \quad (i, j = 1, \ldots, n), \tag{6}$$

is the $n \times n$ variance-covariance matrix given by (1.6).

3 The Ex Post Model

An ex post realization of the standard model requires two ingredients. First, we need a historical sampling of m periodic returns for each of the n-securities. Then the random row matrix $R = [r_1, \ldots, r_n]$ is replaced with the $m \times n$ matrix of historical returns:

$$R = [\mathbf{r}_1, \ldots, \mathbf{r}_n] = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \ldots & r_{mn} \end{bmatrix}, \tag{7}$$

and equation (2) is replaced by the matrix equation

$$\mathbf{r} = R\,\mathbf{p}\,. \tag{8}$$

Now the m-vector $\mathbf{r}$ contains the historical returns of the portfolio $\mathbf{p}$.

Second, we need an a priori system of positive weights, one for each period,

$$\boldsymbol{\mu} = [\mu_1, \ldots, \mu_m]^T, \quad \sum_{k=1}^{m} \mu_k = 1, \quad \mu_k > 0 \ (k = 1, \ldots, m), \tag{9}$$

to compute expected values. Typically $\mu_k = \frac{1}{m}$ for $k = 1, \ldots, m$, but, for investment purposes, one might wish to weight recent periods more heavily than those further in the past. Now the matrix of expected returns (4) is replaced with

$$E = [\bar{r}_1, \ldots, \bar{r}_n] = \boldsymbol{\mu}^T R\,, \tag{10}$$

with the ex post expected return $\bar{r}$ of a portfolio $\mathbf{p}$ still given by equation (3).

Variance measures how actual returns deviate from expected return. The deviations for each security are contained in the deviation vectors

$$\mathbf{z}_j = \mathbf{r}_j - \mathbf{1}\bar{r}_j \ (j = 1, \ldots, n), \tag{11}$$

where $\mathbf{1}$ represents the m-vector that is constantly 1. These deviation vectors have no expected return since

$$\boldsymbol{\mu}^T \mathbf{z}_j = \boldsymbol{\mu}^T \mathbf{r}_j - \boldsymbol{\mu}^T \mathbf{1}\bar{r}_j = \bar{r}_j - \bar{r}_j = 0\,.$$

In this sense they represent pure risk. The variance of $\mathbf{r}_j$ is its μ-average square deviation from expected return:

$$v_{jj} = \sum_{k=1}^{m} \mu_k z_{kj}^2 = \sum_{k=1}^{m} \mu_k (r_{kj} - \bar{r}_j)^2. \tag{12}$$

The covariance of $\mathbf{r}_i$ and $\mathbf{r}_j$ is the μ-average product of the respective deviations:

$$v_{ij} = \sum_{k=1}^{m} \mu_k z_{ki} z_{kj} = \sum_{k=1}^{m} \mu_k (r_{ki} - \bar{r}_i)(r_{kj} - \bar{r}_j)\,. \tag{13}$$

The variance-covariance matrix $V = [v_{ij}]$ can be expressed as

$$V = \begin{bmatrix} \mathbf{z}_1^T \\ \mathbf{z}_2^T \\ \vdots \\ \mathbf{z}_n^T \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & \ldots & 0 \\ 0 & \mu_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_m \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 & \mathbf{z}_2 & \ldots & \mathbf{z}_n \end{bmatrix}, \tag{14}$$

and the ex post variance v of a portfolio $\mathbf{p}$ continues to be given by equation (5).

The standard deviation of the sample vector $\mathbf{r}_j$ is the square root of its variance

$$\sigma_j = \sqrt{v_{jj}}, \tag{15}$$

and the correlation of $\mathbf{r}_i$ and $\mathbf{r}_j$ is

$$c_{ij} = \frac{v_{ij}}{\sigma_i \sigma_j}. \tag{16}$$

Thus the correlation matrix $C = [c_{ij}]$ is related to the variance-covariance matrix V by the matrix equation

$$V = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_m \end{bmatrix} C \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_m \end{bmatrix}. \tag{17}$$

4 The Portfolio Problem

So much for background. Now let's get to the point. The ex post standard model has certain problems that we will illustrate by examining a specific investment portfolio in four exchange traded funds over the 65-week period from Friday, 29-Sep-2006, through Friday, 28–Dec–2007. The funds are

`IVV` – iShares S&P 500 Index Fund
`IWM` – iShares Russell 2000 Index Fund
`EFA` – iShares MSCI EAFE Index Fund
`EEM` – iShares MSCI Emerging Markets Index Fund

We start with an investment of $100,000 in these funds on 29-Sep-2006, allocated as shown at the top of Table 1. Distributions are reinvested automatically. We simply let our money ride for five quarters.

This is how our investment portfolio grows.

Table 1: One investment portfolio at three Friday closings

Friday, 29-Sep-2006			
IVV	$41,119	41.119%	307.431 shares @ 133.75
IWM	$20,155	20.155%	279.933 shares @ 72.00
EFA	$24,847	24.847%	366.739 shares @ 67.75
EEM	$13,879	13.879%	143.426 shares @ 96.77
portf	$100,000	100.000%	
Friday, 29-Dec-2006			
IVV	$43,872	40.000%	308.957 shares @ 142.00
IWM	$21,936	20.000%	281.122 shares @ 78.03
EFA	$27,420	25.000%	374.487 shares @ 73.22
EEM	$16,452	15.000%	144.100 shares @ 114.17
portf	$109,680	100.000%	
Friday, 28-Dec-2007			
IVV	$46,463	38.544%	314.787 shares @ 147.60
IWM	$21,731	18.027%	283.838 shares @ 76.56
EFA	$30,337	25.166%	384.059 shares @ 78.99
EEM	$22,015	18.263%	144.712 shares @ 152.13
portf	$120,545	100.000%	

So what portfolio proportions $p_{\texttt{IVV}}, p_{\texttt{IWM}}, p_{\texttt{EFA}}, p_{\texttt{EEM}}$ might we put into the ex post Markowitz model? The proportionate values of the securities in the investment portfolio vary from one day to the next as prices change and dividends are reinvested. Table 1 just shows the proportions on three different Fridays.

We refer to this multiple proportion dilemma as the "portfolio problem." What is a portfolio anyhow? What are the proportions $\mathbf{p}$ of the securities in a portfolio?

5 The Return Problem

Markowitz's theory is based on returns. What are returns? Where do you get them?

The returns of a security depend on two ingredients—closing prices and distributions. Given sequences of closing prices $\{C_i\}$ and distributions $\{D_i\}$ for a particular security, indexed by successive business days i, we refer to any sequence of positive numbers $\{f_i\}$ as *adjusted closing prices* for the security if

$$\frac{f_i}{f_{i-1}} = \frac{C_i}{C_{i-1} - D_i} \qquad \text{for every business day } i \text{ after the first business day.}$$

Given two business days, i_1 and i_2, with day i_1 earlier than day i_2, the ratio f_{i_2}/f_{i_1} measures the growth of the security (one plus total return) from the close of day i_1 to the close of day i_2. Thus, any sequence of adjusted closing prices encodes the complete return history of the security for the period under consideration. (For more about adjusted closing prices see Norton [2010].)

In Table 6 of Appendix A we give Friday adjusted closing prices for the securities and the investment portfolio of the last section over the same 65 weeks. Closing-price, distribution, and adjusted-closing-price data for all 314 business days in this period are contained in

`http://vic.norton.name/finance-math/probmark/adjclose.csv`

The adjusted closing prices of our investment portfolio come from the matrix equation

$$\mathbf{f}_{\texttt{portf}} = \mathbf{f}_{\texttt{IVV}} \times s_{\texttt{IVV}} + \mathbf{f}_{\texttt{IWM}} \times s_{\texttt{IWM}} + \mathbf{f}_{\texttt{IWM}} \times s_{\texttt{EFA}} + \mathbf{f}_{\texttt{IWM}} \times s_{\texttt{EEM}}\,, \tag{18}$$

where the *nominal portfolio* $(s_{\texttt{IVV}}, s_{\texttt{IWM}}, s_{\texttt{EFA}}, s_{\texttt{EEM}})$ is (0.40, 0.20, 0.25, 0.15), and the $\mathbf{f}_j$ correspond to columns of Table 6. Since the adjusted closing prices of the securities are normalized at \$100 on 29-Dec-2006, the adjusted closing price of the investment portfolio is \$100 on this day as well.

Think of the s_j as notional shares of the securities in the investment portfolio. The shares are fixed for the duration. The proportions of the securities in the portfolio at the close of day i are

$$p_{ij} = \frac{f_{ij}}{f_{i,\texttt{portf}}} \times s_j\,, \quad j \in \{\texttt{IVV}, \texttt{IWM}, \texttt{EFA}, \texttt{EEM}\}\,. \tag{19}$$

These proportions p_{ij} equal the shares s_j *only* on day i = 29-Dec-2006. This, of course, is the portfolio problem—once again.

Let the rows of the adjusted closing prices in Table 6 be indexed by integers starting from 0. Thus $i = 0 \sim$ 2007-12-28, $\ldots, i = 65 \sim$ 2006-09-29. Then the 13-week compound returns terminating on the 53 Fridays from 29-Dec-2006 to 28-Dec-2007, inclusive, are given by

$$r_{ij} = \frac{f_{ij}}{f_{i+13,j}} - 1\,, \quad i = 0, \ldots, 52\,,\ j \in \{\texttt{IVV}, \texttt{IWM}, \texttt{EFA}, \texttt{EEM}, \texttt{portf}\}, \tag{20}$$

and the 13-week continuous returns terminating on the same Fridays by

$$r_{ij} = \ln\left(\frac{f_{ij}}{f_{i+13,j}}\right), \quad i = 0, \ldots, 52\,, \; j \in \{\texttt{IVV}, \texttt{IWM}, \texttt{EFA}, \texttt{EEM}, \texttt{portf}\}. \qquad (21)$$

The percentage rates, for every fourth Friday, are shown in Table 2 and Table 4, respectively.

First consider Table 2. Certainly the ex post Markowitz model should apply here. The returns on the securities make up the matrix R, and $\mathbf{r}$ is the column of investment portfolio returns. However, there is no portfolio $\mathbf{p}$ that satisfies hypothesis (8), $\mathbf{r} = R\mathbf{p}$, of the ex post model—that is to say, no *theoretical* portfolio $\mathbf{p}$ in the four securities can possibly reproduce the actual 13-week returns of our investment portfolio. The ex post Markowitz model simply does not work for this example.

Table 2: Returns – 13-week compound rates

Friday	IVV	IWM	EFA	EEM	portf
2007-12-28	-2.997	-4.025	-1.914	2.225	-2.000
2007-11-30	1.221	-2.418	5.815	15.267	4.003
2007-11-02	5.716	5.804	8.862	26.301	9.873
2007-10-05	2.347	-0.549	2.048	13.572	3.532
2007-09-07	-2.791	-6.001	-2.950	4.798	-2.298
2007-08-10	-3.692	-5.018	-4.428	1.270	-3.377
2007-07-13	7.084	4.551	6.229	16.259	7.771
2007-06-15	10.928	9.346	10.294	20.032	11.788
2007-05-18	5.023	0.882	5.184	7.509	4.604
2007-04-20	4.346	6.054	8.671	9.339	6.508
2007-03-23	2.355	4.258	5.692	4.439	3.879
2007-02-23	4.150	4.758	9.832	8.295	6.269
2007-01-26	3.609	2.970	6.436	11.492	5.285
2006-12-29	6.695	8.835	10.357	18.535	9.680
	best fitting portfolio				error
	0.4112	0.1844	0.2484	0.1561	1.470%

The theoretical portfolio $\mathbf{p}$ that best fits $\mathbf{r} = R\mathbf{p}$, in the least squares sense, appears at the bottom of Table 2. Its proportions are quite different from the nominal proportions, (0.40, 0.20, 0.25, 0.15), that were used to construct the investment portfolio. This $\mathbf{p}$ is determined by the Lagrange multiplier equations

$$R^T(\mathbf{r} - R\mathbf{p}) = \mathbf{1}\lambda, \quad \mathbf{1}^T\mathbf{p} = 1,$$

where $\mathbf{1} = [1, 1, 1, 1]^T$. The relative error of approximation is $\|\mathbf{r} - R\mathbf{p}\|_2 / \|\mathbf{r}\|_2 = 1.470\%$. In Appendix C we give an efficient numerical method for computing the best-fitting $\mathbf{p}$.

Table 3: 13-week compound rate statistics

stat	IVV	IWM	EFA	EEM	portf
mean	3.142	2.103	5.009	11.381	4.680
gmean	3.064	1.987	4.897	11.160	4.583

Table 3 shows two statistics for the returns in Table 2. The means seem meaningless to us: they are unrelated to the appropriate exponential model. The "gmeans" are the 13-week compounds returns that best fit the return data. They are defined by

$$\text{gmean}\{r_i : i = 1, \ldots, m\} = \sqrt[m]{\prod_{i=1}^{m}(1 + r_i)} - 1\,. \tag{22}$$

Appendix B expands on this point.

So far we have only considered a single investment portfolio. In fact, adjusted closing prices for every possible investment portfolio in the four securities can be generated by equation (18) with the s_j satisfying

$$\sum s_j = 1, \quad s_j \geq 0, \quad j \in \{\texttt{IVV}, \texttt{IWM}, \texttt{EFA}, \texttt{EEM}\}. \tag{23}$$

There are 176,851 such investment portfolios when the s_j are restricted to having only two decimal places. If we leave the security returns of Table 2 as they are, but replace the `portf` column by the 176,851 vectors of investment portfolio returns, we find that hypothesis (8), $\mathbf{r} = R\mathbf{p}$, can be satisfied only when the investment portfolio consists of just one security. In the remaining 176,847 cases, the best fitting theoretical returns miss the actual investment returns by an average of 1.413% (compared to the error of 1.470% in our example) with a standard deviation of 0.501%.

This is what we call the "return problem" of the ex post version of Markowitz's standard model—theoretical portfolios virtually never produce investment returns when compound returns are used. Yet compound returns, e.g., quarterly returns, are what one gets from almost every source. Compound returns are characteristically what Markowitz's mean-variance analysis is applied to.

The continuous 13-week rates in Table 4 (next page) look pretty much like the compound rates in Table 2, except the positive rates are a bit smaller and the negative rates larger. Again, there is no possibility of satisfying hypothesis (8), $\mathbf{r} = R\mathbf{p}$, but the mean rates in Table 5 do have meaning now. They are the 13-week continuous rates that best approximate the return data, and the standard deviations measure how much the data varies from these mean rates. (See Appendix B.)

Table 4: Returns – 13-week continuous rates

Friday	IVV	IWM	EFA	EEM	portf
2007-12-28	-3.043	-4.109	-1.933	2.201	-2.020
2007-11-30	1.214	-2.447	5.652	14.208	3.925
2007-11-02	5.559	5.642	8.491	23.349	9.415
2007-10-05	2.319	-0.551	2.027	12.727	3.471
2007-09-07	-2.830	-6.189	-2.994	4.686	-2.325
2007-08-10	-3.762	-5.148	-4.529	1.262	-3.436
2007-07-13	6.844	4.450	6.043	15.065	7.484
2007-06-15	10.372	8.935	9.798	18.259	11.144
2007-05-18	4.901	0.879	5.054	7.240	4.501
2007-04-20	4.254	5.878	8.316	8.928	6.305
2007-03-23	2.327	4.170	5.536	4.344	3.806
2007-02-23	4.066	4.648	9.378	7.969	6.081
2007-01-26	3.546	2.927	6.237	10.878	5.150
2006-12-29	6.481	8.467	9.855	17.004	9.239
	best fitting portfolio				error
	0.4029	0.1835	0.2491	0.1645	1.706%

Table 5: 13-week continuous rate statistics

stat	IVV	IWM	EFA	EEM	portf
mean	3.018	1.968	4.781	10.580	4.481
stdev	3.916	4.779	4.654	6.283	4.308

We can stop our computations here. What is the rationale for computing covariances and correlations when hypothesis (8) is false?

6 Conclusion

In this paper we have tried to point out two problems that may occur in ex post implementations of Harry Markowitz's standard mean-variance model for portfolio choice. The "portfolio problem" can be overcome by fiat. Portfolio proportions might always refer to proportions at the close of a specific business day. The "return problem" is more serious. Compound returns, continuous or not, invariably violate hypothesis (8), $\mathbf{r} = R\mathbf{p}$, of the ex post model—the model simply will not work with monthly or quarterly returns. The solution of this return problem is to use *linear* returns in Markowitz's *linear* model. We'll say more about this in a subsequent paper.

A Friday Adjusted Closing Prices

These Friday adjusted closing have been normalized at 100.000 on Friday, 29-Dec-2006. The adjusted closing prices for Good Friday, 6-Mar-2007, are those of the day before.

Table 6: Adjusted closing prices for four ETFs and an investment portfolio

	nominal portfolio				
	0.40	0.20	0.25	0.15	1.00
Friday	IVV	IWM	EFA	EEM	portf
2007-12-28	105.905	99.064	110.638	133.814	109.906
2007-12-21	106.372	101.362	109.615	134.641	110.421
2007-12-14	105.130	96.719	109.178	131.295	108.385
2007-12-07	107.792	101.220	114.122	140.168	112.917
2007-11-30	106.165	98.847	113.330	135.237	110.853
2007-11-23	103.039	96.797	110.762	129.088	107.629
2007-11-16	104.345	98.383	111.063	134.711	109.387
2007-11-09	103.753	99.105	111.623	134.536	109.408
2007-11-02	108.085	102.561	115.761	141.324	113.885
2007-10-26	109.841	105.759	116.034	142.770	115.512
2007-10-19	107.235	101.994	112.251	133.310	111.352
2007-10-12	111.839	108.094	116.020	139.704	116.315
2007-10-05	111.439	108.493	114.996	137.576	115.660
2007-09-28	109.177	103.219	112.797	130.901	112.149
2007-09-21	108.600	104.152	110.216	127.004	110.875
2007-09-14	105.992	100.460	106.924	120.145	107.242
2007-09-07	104.017	99.997	105.613	115.748	105.372
2007-08-31	104.884	101.296	107.102	117.325	106.587
2007-08-24	105.544	102.441	106.883	114.741	106.638
2007-08-17	103.022	100.871	101.912	107.077	102.923
2007-08-10	102.930	100.421	104.657	111.719	104.178
2007-08-03	102.241	96.935	106.337	111.895	103.652
2007-07-27	103.328	98.581	106.037	114.785	104.774
2007-07-20	109.275	107.098	112.551	122.607	111.658
2007-07-13	110.198	109.221	114.122	125.077	113.215
2007-07-06	108.884	109.092	112.688	121.135	111.714
2007-06-29	106.937	106.648	110.311	115.293	108.976
2007-06-22	106.933	106.470	109.219	115.127	108.641
2007-06-15	108.843	108.281	110.339	115.985	110.176
2007-06-08	107.003	106.381	108.823	110.449	107.851
2007-06-01	109.083	108.756	111.390	113.979	110.329

table 6 continued on next page

Friday	IVV	IWM	EFA	EEM	portf
2007-05-25	107.506	105.918	109.519	110.143	108.087
2007-05-18	108.093	104.942	109.724	111.106	108.323
2007-05-11	106.876	105.726	109.506	110.318	107.820
2007-05-04	106.947	106.342	109.997	109.249	107.934
2007-04-27	106.006	105.957	108.509	107.927	106.910
2007-04-20	105.242	105.687	109.369	107.874	106.758
2007-04-13	102.908	104.467	107.430	107.585	105.052
2007-04-06	102.243	103.286	106.173	105.176	103.874
2007-03-30	100.687	102.015	104.207	102.041	102.036
2007-03-23	101.634	103.029	104.480	102.409	102.741
2007-03-16	98.120	99.026	100.041	96.628	98.558
2007-03-09	99.296	100.192	101.202	98.275	99.799
2007-03-02	97.937	98.501	98.880	94.596	97.784
2007-02-23	102.577	105.024	104.725	102.917	103.654
2007-02-16	102.923	104.024	104.316	103.346	103.555
2007-02-09	101.585	102.922	101.939	99.982	101.700
2007-02-02	102.162	102.845	102.035	101.034	102.098
2007-01-26	100.331	100.051	100.219	99.142	100.069
2007-01-19	100.859	99.654	100.642	98.660	100.234
2007-01-12	101.106	101.217	99.836	98.012	100.347
2007-01-05	99.232	98.334	98.607	96.549	98.494
2006-12-29	100.000	100.000	100.000	100.000	100.000
2006-12-22	99.296	98.821	98.853	98.056	98.904
2006-12-15	100.382	100.687	99.014	99.035	99.899
2006-12-08	99.232	100.534	97.797	97.553	98.882
2006-12-01	98.398	99.258	96.781	95.330	97.706
2006-11-24	98.490	100.254	95.350	95.034	97.539
2006-11-17	98.553	100.292	95.189	93.273	97.268
2006-11-10	97.067	97.650	95.256	92.392	96.030
2006-11-03	95.792	95.494	93.838	89.969	94.370
2006-10-27	96.836	97.165	94.159	88.923	95.046
2006-10-20	96.016	96.476	93.343	88.748	94.350
2006-10-13	95.897	96.859	92.354	88.836	94.145
2006-10-06	94.727	93.796	91.110	86.220	92.361
2006-09-29	93.725	91.882	90.615	84.363	91.175

end of table 6

B The Exponential Model

Assume the value of a security grows according to the exponential model

$$x(t) = x_0 e^{(r/\Delta)(t-t_0)}. \tag{24}$$

Here r is the continuous rate of growth per Δ units of time. ($\Delta = 13$ weeks in our example.) Sampling values at various times t_k and $t_k - \Delta$ $(k = 1, \dots, m)$,

$$x_k \approx x_0 e^{(r/\Delta)(t_k-t_0)},$$
$$y_k \approx x_0 e^{(r/\Delta)(t_k-\Delta-t_0)},$$

leads to the estimates

$$r \approx r_k = \ln\left(\frac{x_k}{y_k}\right). \tag{25}$$

Given a sequence of positive weights $\{\mu_k : k = 1, \dots, m\}$ that sum to 1 as in (9), the μ-mean return

$$\bar{r} = \sum_{k=1}^{m} \mu_k r_k \tag{26}$$

best approximates the rate data r_k in the sense that it minimizes the weighted sum of square deviations

$$r \mapsto \sum_{k=1}^{m} \mu_k (r_k - r)^2.$$

The minimum value

$$v = \sum_{k=1}^{m} \mu_k (r_k - \bar{r})^2 \tag{27}$$

is the μ-variance of the sample rates r_k.

On the other hand, when we are working with compound Δ-rates,

$$r_k = \frac{x_k}{y_k} - 1, \quad k = 1, \dots, m, \tag{28}$$

the expressions (26) and (27) in the r_k are not minimal in any sense that we know of. However we can construct a compound Δ-rate $\hat{r}$ that most closely approximates the r_k by working in the continuous mode. Start with continuous rates $s_k = \ln(1 + r_k)$ corresponding to the compound rates r_k of (28). The best approximating continuous rate is $\bar{s} = \sum_{k=1}^{m} \mu_k s_k$. Then convert $\bar{s}$ to the compound rate $\hat{r} = e^{\bar{s}} - 1$ or

$$\hat{r} = \prod_{k=1}^{m} (1 + r_k)^{\mu_k} - 1. \tag{29}$$

This $\hat{r}$ is the compound Δ-rate that best approximates the r_k of (28) in the sense that

$$r \mapsto \sum_{k=1}^{m} \mu_k (\ln(1 + r_k) - \ln(1 + r))^2$$

is minimal at $r = \hat{r}$. Moreover, when $\mu_k = 1/m$ for $k = 1, \ldots, m$, we can express $\hat{r}$ as

$$\hat{r} = \text{gmean}(r_1, \ldots, r_m) = \sqrt[m]{\prod_{k=1}^{m} (1 + r_k)} - 1. \tag{30}$$

C Octave Code

This is an Octave function <http://www.gnu.org/software/octave/> for computing the portfolio $\mathbf{p}$ that minimizes $\|\mathbf{r} - R\mathbf{p}\|_2$.

```
function p = minnormsum1(R, r)

## Minimize norm(r - R * p, 2) subject to sum(p) = 1.
## We assume rank(R) = columns(R).
## This is an implementation of Algorithm 12.1.2,
## Golub & Van Loan, Matrix Computations, 2nd Edition,
## Johns Hopkins U Press, 1989.

  n = columns(R);
  sqrtn = sqrt(n);

  ## Householder reflection
  ## H: [1, 1, ..., 1]' -> [sqrtn, 0, ..., 0]'
  v = ones(n, 1);   v(1) -= sqrtn;
  H = eye(n) - v * (v' / (n - sqrtn));

  ## Modified Algorithm 12.1.2
  y = 1/sqrtn;
  R = R * H;
  z = R(:, 2 : n) \ (r - R(:, 1) * y);
  p = H * [y; z];

endfunction
```

References

Harry M. Markowitz. *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. Blackwell, 1987.

Vic Norton. Adjusted closing prices. <http://vic.norton.name/finance-math/adjustedClosingPrices.pdf>. Unpublished manuscript, May 2010.