Notional prices and Markowitz's portfolio selection model

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ABSTRACT

Markowitz's portfolio selection model posits a linear relationship between the return of a portfolio and the returns of its component securities. This linear relationship does not hold in an ex post setting when sample returns are computed in the usual way.

We propose using a priori notional price weights to determine ex post notional prices. Then the corresponding linear returns do satisfy the hypotheses of Markowitz's model, and the meaning of a portfolio, as a list of security proportions, is unambiguous.

1 The Standard Portfolio Selection Model

In the mean-variance portfolio selection model of Markowitz [1987, pg. 1]

> an investor is to choose fractions $p_1, p_2, \ldots, p_n$ invested in n securities subject to constraints
>
> $$\sum_{j=1}^{n} p_j = 1, \qquad p_j \geq 0 \quad (j = 1, \ldots, n)\,. \tag{M1.1}$$
>
> We suppose that the returns this period on individual securities $r_1, r_2, \ldots, r_n$ are jointly distributed random variables, and the return on the portfolio is
>
> $$R = \sum_{j=1}^{n} r_j p_j\,. \tag{M1.2}$$
>
> The expected (mean) return of the portfolio as a whole is
>
> $$E = E(R) = \sum_{j=1}^{n} e_j p_j\,, \tag{M1.3}$$
>
> where
>
> $$e_j = E(r_j) \quad (j = 1, \ldots, n) \tag{M1.4}$$
>
> and the E is the expected value operator.

This is *almost* an exact quote. We have changed some of the symbols and the order of multiplication to suit our purposes. Most importantly, the middle expression, $E(R)$, has been inserted in (M1.3) to help make the point of this article: that equations (M1.2) and (M1.3) are not generally valid in an ex post setting with the usual method of computing returns.

2 The Ex Post Return Problem

What are returns, anyhow? How do you compute them? Let's look at an example.

Suppose we invest \$100,000 in securities A and B with 40% going to A and 60% to B. Suppose A earns 6% each year for two consecutive years and B earns 9% each year. If the expected return function is the average of the two yearly returns, then clearly

$$e_A = E(r_A) = 6\% \quad \text{and} \quad e_B = E(r_B) = 9\%.$$

Thus the right hand side of (M1.3) is

$$e_A p_A + e_B p_B = 6\% \times 40\% + 9\% \times 60\% = 7.80\%.$$

Now let us compute the ex post expected return of the actual portfolio. We start with \$100,000 and put 40% in A and 60% in B. If A is selling at \$50 per share and B sells for \$30, we will buy 800 shares of A (for \$40,000) and 2,000 shares of B (for \$60,000). The price (and value) of A increases by 6% each year and B by 9%, so the prices per share are

year	A	B
0	50.0000	30.0000
1	53.0000	32.7000
2	56.1800	35.6430

Table 1: Prices

Multiplying these prices per share by the 800 and 2,000 shares of A and B, respectively, gives us the yearly values of our holdings

year	A	B	P
0	\$40,000	\$60,000	\$100,000
1	\$42,400	\$65,400	\$107,800
2	\$44,944	\$71,286	\$116,230

Table 2: Values

with the portfolio values, the sums of the A and B values, in the P-column.

Returns for the two years would typically be computed as

$$r_{ij} = \frac{a_{ij}}{a_{i-1,j}} - 1 \qquad (i = 1, 2; \quad j = A, B, P)\,, \tag{1}$$

where the a_{ij} are the values in Table 2. These returns, r_{ij}, are shown in the first two rows of Table 3. The expected return row is the average of the first two rows.

year	r_A	r_B	r_P
1	6.00%	9.00%	7.80%
2	6.00%	9.00%	7.82%
expected return	6.00%	9.00%	7.81%

Table 3: Returns

Thinking of the first two rows of Table 3 as vectors of returns, the vector equation

$$\mathbf{r}_P = \mathbf{r}_A p_A + \mathbf{r}_B p_B \tag{2}$$

is clearly false; that is to say equation (M1.2) is false in this ex post framework. The expected return, $e_P = E(\mathbf{r}_P) = 7.81\%$, of the portfolio P corresponds to the middle, $E(R)$ term of equation (M1.3). Since

$$7.81\% = E(R) \neq \sum_{j=A,B} e_j p_j = 7.80\%$$

equation (M1.3) is false as well.

The deviations of the sample returns of the portfolio from its expected return are $\mp 0.01\%$, so the standard deviation of return is $\sigma_P = 0.01\%$. In effect P is a risky portfolio composed of two risk-free securities. This is not supposed to happen in the standard model.

3 The Error Of Assumed Associativity

The failure of the (M1.3) assumption of the standard model (in our ex post interpretation) is caused by a lack of associativity. We are implementing three (ex post) processes, **P, R,** and **E**, in this order:

P Invest in a portfolio of securities. The proportions invested in each security make up the portfolio $P: p_1, \ldots p_n$.

R Compute periodic "returns."

E Compute expected values (of returns).

The right-hand, $\sum_{j=1}^{n} e_j p_j$, part of equation (M1.3) corresponds to an (**ER**)**P** association—combine the expected returns of the individual securities in the proportions of portfolio P. The middle, $E(R)$, part of (M1.3) corresponds to an **E**(**RP**) association—the expected return of the portfolio P is the expected value of its periodic returns.

These two ways of associating **E, R** and **P** are not generally equivalent. The relative error resulting from the associativity implicitly assumed in (M1.3) is

$$\text{error} = (\mathbf{ER})\mathbf{P}/\mathbf{E}(\mathbf{RP}) - 1 = 7.80/7.81 - 1 = -0.1283\%$$

in our example.

4 Things Can Get Worse

Samples of two returns (and three prices) are a bit small for statistical purposes. Things might get better if we considered larger samples, say two years of monthly returns. Unfortunately the error of assumed associativity just gets worse.

To see this we'll just refine the previous example. Now suppose that A earns 0.50% each month for 24 consecutive months and B earns 0.75% per month. Again, our portfolio consists of 40% of A and 60% of B at \$50 and \$30 per share, respectively, at the beginning of the two-year run. The computations proceed as before. In the end

$$\begin{aligned} e_A &= E(r_A) = 0.5000\%, \\ e_B &= E(r_B) = 0.7500\%, \\ e_P &= E(r_P) = 0.6517\%. \end{aligned}$$

Since $e_A p_A + e_B p_B = 0.6500\%$, the error of assumed associativity is

$$\text{error} = (\mathbf{ER})\mathbf{P}/\mathbf{E}(\mathbf{RP}) - 1 = 0.6500/0.6517 - 1 = -0.2620\%,$$

roughly twice what it was before.

5 Portfolios and Time

The definition of a portfolio seems so simple: the proportions $p_1, \ldots, p_n$ invested in n securities subject to the constraints (M1.1). However a real portfolio varies from one time to the next as the prices of its component securities vary. The number of shares held in each security is much less transitory than the proportionate value of these shares.

Consider the two-security, two-year example of section 2. If we rewrite the values (Table 2) in terms of portfolio proportions, we arrive at

year	A	B	value of P
0	40.0000%	60.0000%	\$100,000.00
1	39.3321%	60.6679%	\$107,800.00
2	38.6682%	61.3318%	\$116,230.00
shares	800.0000	2,000.0000	

Table 4: Portfolio 0

This table is labelled "Portfolio 0" because we started with the nominal 40%–60% split at year 0. But there are three "portfolios" here. The year 0 portfolio is the nominal one by fiat.

Suppose an investor puts \$100,000 in securities A and B at the year 0 prices of Table 1, and he wants to reallocate his investment at year 2. To him a 40%–60% reinvestment portfolio would mean a year 2 portfolio. The situation would look like this.

year	A	B	value of P
0	41.3468%	58.6532%	\$100,000.00
1	40.6716%	59.3284%	\$107,759.60
2	40.0000%	60.0000%	\$116,143.13
shares	826.9358	1,955.1070	

Table 5: Portfolio 2

Table 4 and Table 5 both start with a \$100,000 investment and both represent the price data of Table 1. Indeed the Table 1 prices can be recovered from either Table 4 or Table 5 by simply multiplying the P-values by the component percentages and then dividing by the number of shares of the respective securities. Both tables represent 40%–60% portfolios. Time is the difference here. The nominal 40%–60% split depends on security prices at a particular time, either year 0 in the case of Table 4 or year 2 in the case of Table 5.

6 Linear Returns

The ex post return problem with its error of assumed associativity is caused by equation (1). Returns so computed are rates of exponential growth. Exponential (or compound) ex post rates are incompatible with Markowitz's *linear* standard model. We need to work with linear (or simple) rates of return.

To describe the linear rate setup we will formulate the ex post standard model in a somewhat general setting. We start with three parameters,

n – the number of securities,
m – the number of (interest) periods,
d – the length of an (interest) period,

and corresponding adjusted-price data (adjusted for distributions and splits),

f_{ij} – the price of security j at time i ($i = 1, \ldots, m + d; j = 1, \ldots, n$).

We assume the index i increases as time goes backwards so that $i = 1$ corresponds to the most recent time.

The *exponential* d-rates of return corresponding to the given price data are

$$r_{ij} = \frac{f_{ij} - f_{i+d,j}}{f_{i+d,j}} = \frac{f_{ij}}{f_{i+d,j}} - 1\,, \quad (i = 1, \ldots, m; j = 1, \ldots, n). \tag{3}$$

Equation (3) is essentially equation (1) without the $j = P$ part. As we have seen these exponential rates do not work with either hypothesis (M1.2) or (M1.3) of the standard model.

A security earns at a *linear* rate if adjusted-price differences are constant over equal periods of time. However, the specific linear rate of return depends on a reference price.

We propose that the notional reference prices necessary for defining linear rates be determined by a system of a priori

η_i – notional price weights $(i = 1, \ldots, m + d)$

that satisfy $\sum \eta_i = 1$, $\eta_i \geq 0$. Then the *notional price* $\hat{f}_j$ of security j is

$$\hat{f}_j = \sum_{i=1}^{m+d} \eta_i f_{ij} \quad (j = 1, \ldots, n) \tag{4}$$

and the *linear* d-rates of return corresponding to the given price data (and the notional price weights) are

$$r_{ij} = \frac{f_{ij} - f_{i+d,j}}{\hat{f}_j} = \frac{f_{ij}}{\hat{f}_j} - \frac{f_{i+d,j}}{\hat{f}_j}, \quad (i = 1, \ldots, m; j = 1, \ldots, n). \tag{5}$$

These linear rates do work with hypotheses (M1.2) and (M1.3) of the standard model, as we shall see.

First let us firm up these ideas with some specific numbers. We might, for example, want to work with $m = 39$ weeks of $(d = 9)$-week returns. We would start with $m + d = 48$ successive weeks of adjusted weekly closing-prices for each of n securities and compute the $m \times n$ matrix, $[r_{ij}]$, of linear 9-week rates using (5).

The notional price weights η_i $(i = 1, \ldots, m + n)$ must be set in advance. If our goal were to use our results to guide future investment, the most recent prices would be the most important. This would suggest a definition like

$$\eta_i = \begin{cases} \frac{1}{5}, & \text{if } i = 1, \ldots, 5, \\ 0, & \text{if } i > 5, \end{cases}$$

so that the notional price of a security is the average of its last 5 adjusted weekly closing-prices.

Why linear rates work and exponential rates don't work in an ex post version of the standard model is evident from the rate equations (3) and (5). Consider an individual security j. The constant notional price, $\hat{f}_j$, is the divisor in the right-hand side of the first equation in (5). Division by a constant does not affect the linearity of the total expression. On the other hand, $f_{i+d,j}$ is the divisor in the right-hand side of the first equation in (3). $f_{i+d,j}$ does vary with i. This makes the (3)-expression nonlinear in i, and it kills the linear assumptions (M1.2) and (M1.3) of the standard model.

Let us elaborate on why linear rates do work. Start with a portfolio P : $p_1, \ldots, p_n$ at notional prices. The number of shares of security j, per notional dollar invested in the portfolio, is $\hat{s}_j = p_j/\hat{f}_j$. It follows that the value of the portfolio at time i, per notional dollar invested, is

$$\hat{v}_{iP} = \sum_{j=1}^{n} f_{ij}\hat{s}_j = \sum_{j=1}^{n} \frac{f_{ij}}{\hat{f}_j} p_j$$

The notional value of the portfolio itself is, more or less by definition,

$$\begin{aligned}
\hat{v}_P &= \sum_{i=1}^{m+d} \eta_i \hat{v}_{iP} = \sum_{i=1}^{m+d} \eta_i \sum_{j=1}^{n} \frac{f_{ij}}{\hat{f}_j} p_j \\
&= \sum_{j=1}^{n} \frac{\sum_{i=1}^{m+d} \eta_i f_{ij}}{\hat{f}_j} p_j = \sum_{j=1}^{n} \frac{\hat{f}_j}{\hat{f}_j} p_j = \sum_{j=1}^{n} p_j \\
&= 1 \ \text{ dollar per notional dollar invested.}
\end{aligned}$$

The linear returns of the portfolio are

$$\begin{aligned}
r_{iP} &= \frac{\hat{v}_{iP} - \hat{v}_{i+d,P}}{\hat{v}_P} = \hat{v}_{iP} - \hat{v}_{i+d,P} \\
&= \sum_{j=1}^{n} \frac{f_{ij}}{\hat{f}_j} p_j - \sum_{j=1}^{n} \frac{f_{i+d,j}}{\hat{f}_j} p_j = \sum_{j=1}^{n} \frac{f_{ij} - f_{i+d,j}}{\hat{f}_j} p_j \\
&= \sum_{j=1}^{n} r_{ij} p_j
\end{aligned}$$

Thus hypothesis (M1.2) of the (ex post) standard model,

$$\mathbf{r}_P = \sum_{j=1}^{n} \mathbf{r}_j p_j\,, \tag{M1.2}$$

does hold in this case, and equation (M1.3) follows from the linearity of the expected value function.

Remark. The phrase “per notional dollar invested” has been used repeatedly above. It emphasizes the fact that we are not talking about real dollars here. Notional prices are not real prices at some specific time. Notional dollars are needed to invest in notional shares at notional prices.

7 Portfolios and Notional Prices

In section 5 we pointed out that a "portfolio," as a list of proportions, can be quite ambiguous in an ex post setting. The portfolio proportions corresponding to a fixed set of shares vary from one time to the next as the prices of the shares vary. Even if the meaning of "portfolio" is fixed at a particular time, the hypotheses (M1.2) and (M1.3) of the standard model will almost certainly fail to be true when exponential returns (3) are used.

On the other hand, if one adopts a priori notional price weights with the corresponding notional prices (4) and linear returns (5), then portfolio ambiguity disappears. The proportions $p_1, \ldots, p_n$ *must* refer to the proportions of the securities at notional prices, and the notional shares $\hat{s}_j = p_j/\hat{f}_j$ $(j = 1, \ldots, n)$ per notional dollar invested are fixed for the duration, $m + d$, of the ex post model.

For instance if we use Table 1 prices and notional price weights that average year 1 and year 2 prices ($\eta_0 = 0, \eta_1 = 0.5, \eta_2 = 0.5$), then the $p_A = 40\%$ and $p_B = 60\%$ portfolio proportions (at notional prices) produce the following (linear) return table.

year	r_A	r_B	r_P
1	5.4955%	7.9013%	6.9390%
2	5.8252%	8.6124%	7.4976%
expected return	5.6604%	8.2569%	7.2183%

Table 6: (Linear) Returns

Table 6 is comparable to Table 3. Expected return is average return, as in Table 3, but now

$$\mathbf{r}_P = \mathbf{r}_A p_A + \mathbf{r}_B p_B \,. \tag{2}$$

does hold (except for a slight rounding error), and there is no error of assumed associativity.

An investment of \$100,000 in this portfolio, at year 0 prices, will buy

$$s_A = 820.4196 \quad \text{and} \quad s_B = 1,965.9674$$

shares of securities A and B, respectively. The historical values of the portfolio are then

year	A	B	P
0	\$41,020.98	\$58,979.02	\$100,000.00
1	\$43,482.24	\$64,287.13	\$107,769.37
2	\$46,091.17	\$70,072.98	\$116,164.15

Table 7: Values

Table 7 is comparable to Table 2. Both represent a nominal 40%–60% portfolio at Table 1 prices. However, the meaning of a 40%–60% split in the Table 2 case is time dependent, whereas the meaning of a 40%–60% split for Table 7 depends only on the meaning of linear rates of return—as defined by notional prices and a priori notional price weights.

8 Conclusion

Markowitz's standard portfolio selection model posits a linear relationship between the return of a portfolio and the returns of its component securities. This linear relationship does not hold in an ex post setting when returns are computed from adjusted closing prices in the usual way. Indeed it is unclear what a portfolio even means in this setting.

Markowitz's linear model requires linear returns. Ex post linear rates of return depend on the notional prices of the underlying securities. We propose using a priori *notional price weights* to determine notional prices. The hypotheses of the standard model are then satisfied by the resulting ex post linear rates, and what one means by a "portfolio" becomes unambiguous.

References

Harry M. Markowitz. *Mean-Variance Analysis in Portfolio Choice and Capital Markets.* Blackwell, 1987.