A linear algebra exercise

by Vic Norton, 28-Mar-2008

PARAMETERS. n, n_L, n_S, λ,
$n = n_L + n_S,\ n_L \geq 1,\ n_S \geq 1,\ 0 < \lambda < 1.$

NOTATION. $\mathbf{e}_k, \mathbf{f}^\lambda_{ij} \in \mathbb{R}^n$ $(k = 1, \ldots, n;\ i = 1, \ldots, n_L;\ j = 1, \ldots, n_S)$,
$\mathbf{e}_k(s) = \delta_{sk}$ $(s = 1, \ldots, n)$, $\quad \mathbf{f}^\lambda_{ij} = (1 - \lambda)\,\mathbf{e}_i + \lambda\,\mathbf{e}_{n_L+j}$.

SHOW.

1) The $n_L(n_S + 1)$ vectors $\mathbf{e}_i, \mathbf{f}^\lambda_{ij}$ $(i = 1, \ldots, n_L;\ j = 1, \ldots, n_S)$ are convexly independent—that is to say, no one of these vectors can be written as a convex combination of the others.

2) The $n_L(n_S + 1)$ vectors $\mathbf{e}_i, \mathbf{f}^\lambda_{ij}$ $(i = 1, \ldots, n_L;\ j = 1, \ldots, n_S)$ are the vertices of the convex polytope in $\mathbb{R}^n$ defined by the equations and inequalities

$$x_k \geq 0 \ (k = 1, \ldots, n), \quad \sum_{k=1}^{n} x_k = 1, \quad \sum_{k=n_L+1}^{n} x_k \leq \lambda. \tag{1}$$

Every element of this polytope is a convex combination of the $\mathbf{e}_i, \mathbf{f}^\lambda_{ij}$ vectors.

MY INTEREST.

I am interested in portfolios of n securities: n_L is the number of the securities purchased long, n_S the number sold short. The inequality $\sum_{k>n_L} x_k \leq \lambda$ represents the requirement that the total short position be no more than λ percent of the whole.

SKETCH OF PROOF.

Statement 1 is essentially an observation. As for statement 2, it is easy to see that each $\mathbf{x}$ satisfying (1) is uniquely representable as a convex combination of the $\mathbf{e}_i, \mathbf{f}^\lambda_{ij}$ when $n_L = 1$. Furthermore, when $\sum_{k>n_L} x_k = \lambda$, the $\mathbf{f}^\lambda_{ij}$-coefficients of $\mathbf{x}$ can be taken as

$$q_{ij} = \frac{x_i x_{n_L+j}}{(1 - \lambda)\lambda} \tag{2}$$

with the $\mathbf{e}_i$-coefficients zero—though this convex representation is not unique for $n_L > 1$.

In the general case set

$$\mathbf{y} = \mathbf{y}(t) = (1 + t)\mathbf{x} - t\mathbf{e}_1 \tag{3}$$

with $t \geq 0$. Then the vector $\mathbf{y}$ always lies on the $\sum_{k=1}^{n} x_k = 1$ hyperplane. It starts off at $\mathbf{x}$, but, as t increases, it must exit the (1)-polytope at either the $x_1 = 0$ hyperplane or the $\sum_{k>n_L} x_k = \lambda$ hyperplane. If $\mathbf{y}$ exits at the $x_1 = 0$ hyperplane, then, by induction on n_L, the "last" $\mathbf{y}$ can be represented as a convex combination of the $\mathbf{e}_i, \mathbf{f}_{ij}^{\lambda}$ with $i > 1$. On the other hand, if $\mathbf{y}$ exits at the $\sum_{k>n_L} x_k = \lambda$ hyperplane, then the "last" $\mathbf{y}$ is a convex combination of the $\mathbf{f}_{ij}^{\lambda}$ by (2). Writing either "last" $\mathbf{y}$ as

$$\mathbf{y} = \sum_{i=1}^{n_L} p_i \mathbf{e}_i + \sum_{i=1}^{n_L} \sum_{j=1}^{n_S} q_{ij} \mathbf{f}_{ij}^{\lambda} \,,$$

we see from (3) that

$$\mathbf{x} = \left(\frac{p_1 + t}{1 + t}\right) \mathbf{e}_1 + \sum_{i=2}^{n_L} \left(\frac{p_i}{1 + t}\right) \mathbf{e}_i + \sum_{i=1}^{n_L} \sum_{j=1}^{n_S} \left(\frac{q_{ij}}{1 + t}\right) \mathbf{f}_{ij}^{\lambda}$$

is a convex combination of the $\mathbf{e}_i, \mathbf{f}_{ij}^{\lambda}$. □