

How to beat the market and sleep well at night

or

How optimizing a weighted Sharpe ratio
can lead to an effective investment strategy

by

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1. Risk and reward

Let X_A , X_B , and X_M denote the periodic (e.g., quarterly) returns of two risky mutual funds A and B and a money market fund M , respectively. We refer to the differential returns

$$r_A = X_A - X_M \quad \text{and} \quad r_B = X_B - X_M$$

as the *rewards* of funds A and B . The *expected rewards* are

$$\bar{r}_A = \int r_A d\mu \quad \text{and} \quad \bar{r}_B = \int r_B d\mu$$

(where $d\mu$ is the distribution of reward), and the random variables

$$f_A = r_A - \bar{r}_A \quad \text{and} \quad f_B = r_B - \bar{r}_B$$

are the *risks* of A and B .

We will be interested in (formal) portfolios in A and B . Such a portfolio is specified by the respective proportions, $p^A + p^B = 1$, of the funds in the portfolio. The reward, expected reward, and risk of such a portfolio are given by

$$r = r_A p^A + r_B p^B, \quad \bar{r} = \bar{r}_A p^A + \bar{r}_B p^B, \quad \text{and} \quad f = f_A p^A + f_B p^B,$$

respectively. If f_A and f_B are linearly independent, then the expected reward of any portfolio in A and B is a linear function of its risk:

$$\bar{r} = \int e f d\mu,$$

where

$$e = f_A q^A + f_B q^B, \quad q^A = \frac{\begin{vmatrix} \bar{r}_A & \bar{r}_B \\ c_{AB} & c_{BB} \end{vmatrix}}{D}, \quad q^B = \frac{\begin{vmatrix} c_{AA} & c_{AB} \\ \bar{r}_A & \bar{r}_B \end{vmatrix}}{D}, \quad D = \begin{vmatrix} c_{AA} & c_{AB} \\ c_{AB} & c_{BB} \end{vmatrix},$$

and

$$c_{AA} = \int f_A f_A d\mu, \quad c_{AB} = \int f_A f_B d\mu, \quad c_{BB} = \int f_B f_B d\mu.$$

2. The Sharpe ratio

Roughly speaking, the *Sharpe ratio* (Sharpe [1994]) is the ratio of reward to risk. The larger the Sharpe ratio the better. To be more precise, the Sharpe ratio of a portfolio in A and B is the ratio of its expected reward to its *scalar risk* (or standard deviation of reward):

$$s = \bar{r}/\sigma = \frac{\int e f d\mu}{\sqrt{\int f^2 d\mu}}.$$

Rewriting s as

$$s = \sqrt{\int e^2 d\mu} \frac{\int ef d\mu}{\sqrt{\int e^2 d\mu} \sqrt{\int f^2 d\mu}},$$

we see that the Sharpe ratio of the portfolio is

$$s = \|e\| \cos \theta,$$

where $\|e\| = \sqrt{\int e^2 d\mu}$ and $0 \leq \theta \leq \pi$ is the angle between e and f in the $d\mu$ -metric on the risk space spanned by f_A and f_B .

3. The Sharpe-optimal long portfolio

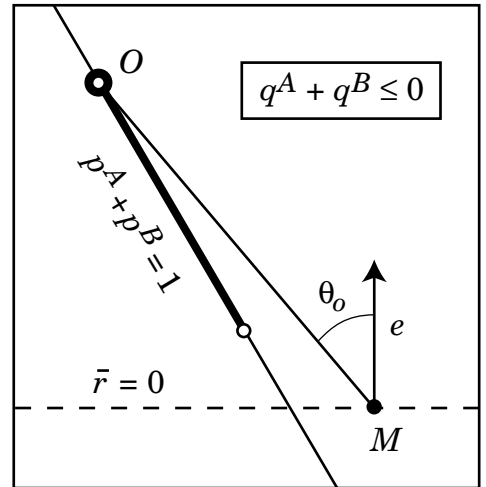
We are looking for an optimal long ($p^A, p^B \geq 0$) portfolio in funds A and B . If $\bar{r}_A \leq 0$ and $\bar{r}_B \leq 0$, then no long portfolio in A and B has positive expected reward. In this case, by definition, money market, M , is optimal. It has zero expected reward and zero risk.

Assume then that either A or B has positive expected reward. The *Sharpe-optimal long portfolio* in A and B is, by definition, the long portfolio O in A and B ($p_O^A + p_O^B = 1$, $p_O^A, p_O^B \geq 0$) of maximal Sharpe ratio. The risk of O , f_O , makes the least angle θ_O with e of all long portfolios, since $s_O = \|e\| \cos \theta_O$.

The following pictures of risk space cover all possible cases. The capital letters, M , O , and E , represent the risk variables, $f_M (= 0)$, f_O , and f_E , respectively.

If $q^A + q^B \leq 0$, then the Sharpe-optimal long portfolio O is either A or B , whichever has the higher expected reward. This is indicated in the picture on the right. The thick segment represents the set of long portfolios in A and B , with A and B at the two extremes.

Money market, M , having zero risk by definition, is the origin of risk space. The vector e is the gradient of expected reward (with respect to the $d\mu$ -metric). It points in the direction of maximum increase of expected reward.

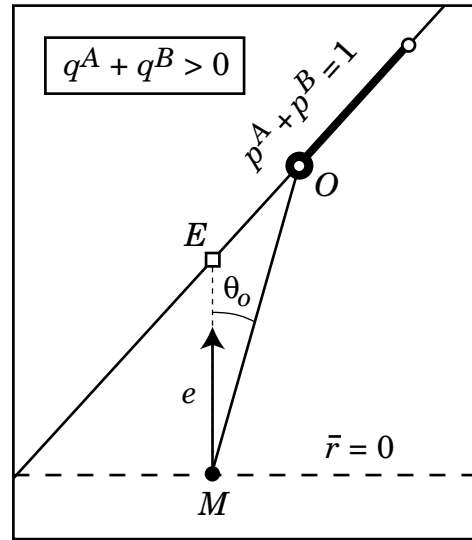
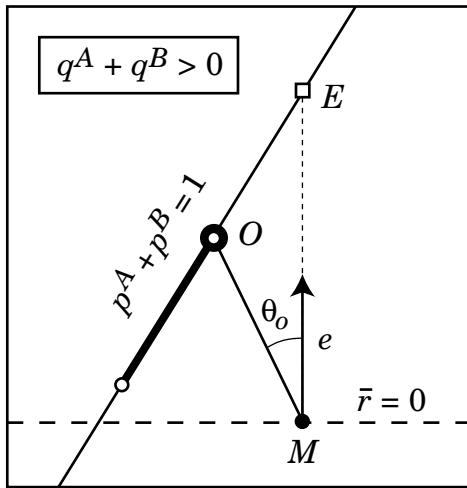
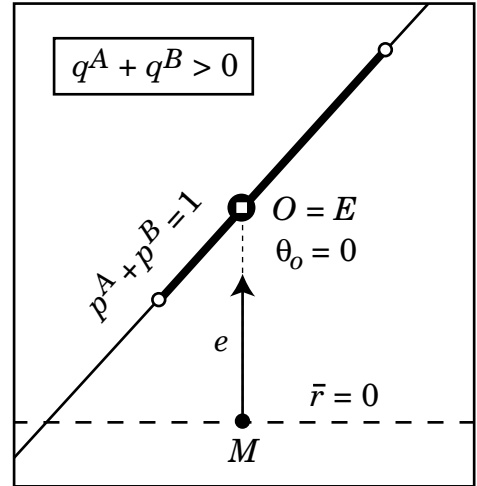


If $q^A + q^B > 0$, there is a unique long-short portfolio E of maximal Sharpe ratio. The A - B proportions in E are

$$p_E^A = \frac{q^A}{q^A + q^B}, \quad p_E^B = \frac{q^B}{q^A + q^B}.$$

When each of these coefficients is nonnegative, as in the picture on the right, the Sharpe-optimal long portfolio is $O = E$.

On the other hand, if one of the coefficients is negative and the other greater than 1, as in the pictures below, then the Sharpe-optimal long portfolio consists entirely of the long fund, the one with the positive E -coefficient—even though this may not be the fund with the highest expected reward.



4. The SOLNG strategy

The *SOLNG* (pronounced “so long”) investment strategy is straightforward. First, fix on two dissimilar, risky funds, A and B , and a money market fund, M . (We would suggest a high-quality bond fund for A , and a broad-based stock fund, perhaps an index fund, for B .) The strategy starts with all moneys invested in M and continues as follows:

1. At the end of a period (e.g., quarter) record the total returns of the three funds for that period.
2. Using the new data and historical returns, estimate the values of \bar{r}_A and \bar{r}_B .

3. If neither \bar{r}_A nor \bar{r}_B is positive, reinvest everything in M . (To simplify notation, *ex ante* estimates carry the labels of their underlying variables.)
4. Otherwise, estimate the values of c_{AA} , c_{AB} , and c_{BB} .
5. Use the formulas at the end of section 1 to estimate the coefficients, q^A and q^B , of e .
6. Compute the Sharpe-optimal long proportions, p_O^A and p_O^B , as described in section 3, and reinvest everything in A and B in these proportions.

That's it. That's the SOLNG strategy. The only thing that remains to be settled is how to compute the *ex ante* estimates of \bar{r}_A , \bar{r}_B , c_{AA} , c_{AB} , and c_{BB} .

5. Weights

Fix a sample size n (e.g., $n = 12$ quarters) and a system of nonnegative weights μ_i ($i = 1, \dots, n$) that sum to 1. Let X_A^i, X_B^i , and X_M^i ($i = 1, \dots, n$) denote the historical returns of A , B , and M over n successive periods, with $i = 1$ corresponding to the latest period. Then the *ex ante* estimates of \bar{r}_A , \bar{r}_B , c_{AA} , c_{AB} , and c_{BB} , corresponding to the n periods under consideration, are computed as

$$\begin{aligned}
 r_A^i &= X_A^i - X_M^i, & r_B^i &= X_B^i - X_M^i, & i &= 1, \dots, n, \\
 \bar{r}_A &= \sum_{i=1}^n \mu_i r_A^i, & \bar{r}_B &= \sum_{i=1}^n \mu_i r_B^i, \\
 f_A^i &= r_A^i - \bar{r}_A, & f_B^i &= r_B^i - \bar{r}_B, & i &= 1, \dots, n, \\
 c_{AA} &= \sum_{i=1}^n \mu_i f_A^i f_A^i, & c_{AB} &= \sum_{i=1}^n \mu_i f_A^i f_B^i, & c_{BB} &= \sum_{i=1}^n \mu_i f_B^i f_B^i.
 \end{aligned}$$

The weights μ_i reflect preconceptions about the distribution $d\mu$. To predict the reward for the next period (e.g., quarter), should the most recent rewards weigh more heavily than those of a couple of years ago? Does recent volatility diminish the certainty of your prediction more than fluctuations of long ago?

If your answer to these questions is no, uniform weights, $\mu_i = 1/n$, should be used. If you believe in momentum and answer yes, then a decreasing system of weights, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, is more appropriate.

6. An example

Let us illustrate these ideas with specific data. We use the quarterly total returns from

- A – Vanguard Total Bond Market Index (VBMFX) (the largest bond index fund),
- B – Vanguard 500 Market Index (VFINX) (the largest stock index fund),
- M – Vanguard Prime Money Market (VMMXX),

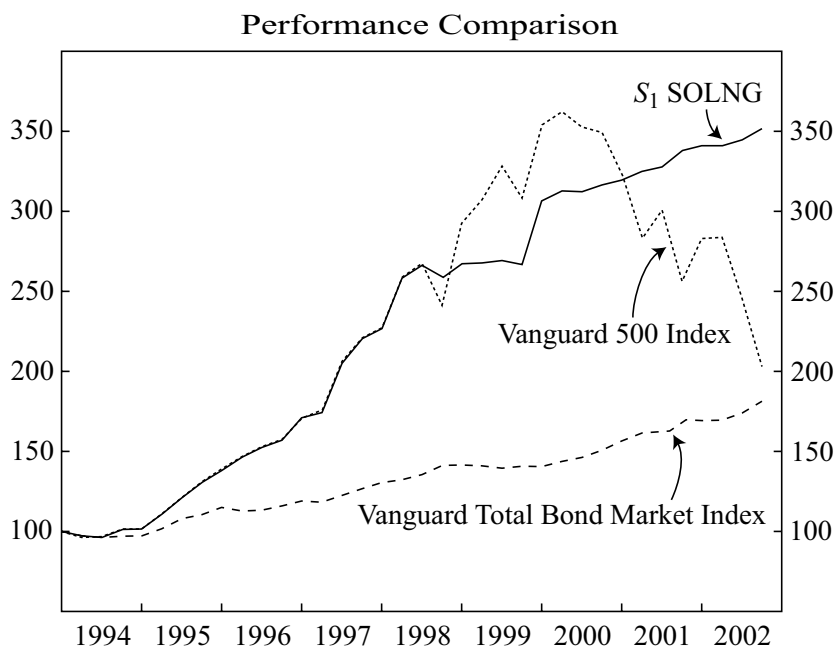
dating from the first quarter of 1991. The returns from the beginning of 1994 are shown in Table 1.

First we examine the results of a specific, 3-year, 12-quarter, SOLNG investment strategy that we label the S_1 strategy. To compute *ex ante* quantities we posit that data of two years ago is only half as important as that from the past year, and three-year-old data counts half as much as two-year-old stuff. Thus the weights for the S_1 strategy are

$$\mu_{1:4} = \frac{1}{7}, \quad \mu_{5:8} = \frac{1}{14}, \quad \mu_{9:12} = \frac{1}{28}.$$

At the end of 1993 the *ex ante* Sharpe-optimal long portfolio corresponding to the S_1 weights consisted of $p_O^A = 70.2\%$ of Total Bond Market and $p_O^B = 29.8\%$ of 500 Index. The S_1 SOLNG strategy starts at the beginning of 1994 with money invested in just these proportions of the two funds.

The picture below shows the performance of the S_1 strategy against the performance of its two risky components.



This picture has two disparate halves. The first half, from the beginning of 1994 through the second quarter of 1998, shows the S_1 SOLNG strategy essentially tracking 500 Index. S_1 was 100% in 500 Index during most of this period. The second half of the picture, the part since 1998q2, shows S_1 and 500 index diverging substantially, with S_1 accumulating more gain at less risk (volatility). S_1 was mostly invested in Total Bond Market during this period (except for 1999q4 and 2000q1, when it shifted back to 100% 500 Index).

This bipartite behavior of the S_1 strategy is a bit mysterious. Here is an intuitive explanation. When 500 Index was producing high *ex ante* expected reward, r_B , with relatively low scalar risk, $\sigma_B = \sqrt{c_{BB}}$, S_1 was content to follow along. The Sharpe ratio of 500 Index, $s_B = r_B/\sigma_B$, was optimal. However, the scalar risk, σ_B , spiked at the end of

1998 to around twice its average over the previous four and a half years and has remained high ever since. Though the expected reward of 500 Index was substantial until the 4th quarter of 2000, S_1 was scared by the increased risk and shifted most of its money into Total Bond Market after 1998.

A second factor is also significant. Stocks and bonds were marching to the same drummer during the 1994–1998q2 period. The *ex ante* (S_1) correlation of 500 Index and Total Bond Market, $\text{cor}(A, B) = c_{AB}/\sigma_A\sigma_B$, was positive throughout this period. This correlation coefficient became negative in the third quarter of 1998 and has remained negative ever since. (See Table 2.)

To complete this section, we compare three other SOLNG strategies with the S_1 strategy. These are

- S_2 : The uniform, 12-quarter strategy, with $\mu_i = 1/12$, $i = 1, \dots, 12$. No momentum here.
- S_3 : The uniform, 4-quarter strategy, with $\mu_i = 1/4$, $i = 1, \dots, 4$. This is more of a momentum strategy than S_1 : at least S_1 believes that two- and three-year-old data count.
- S_4 : The one quarter strategy. Extreme momentum investing. At the beginning of each quarter, reinvest all moneys in whichever fund, A , B , or M , had the highest return over the previous quarter.

The quarterly returns from all four strategies, beginning in 1994, are shown in Table 1.

Summary statistics for the three mutual funds and the four SOLNG strategies, over the almost-nine-year period, are shown in the table below. To conform with common practice, certain statistics have been annualized: annual reward is thought of as the sum of the four quarterly rewards, with the quarterly rewards independent and identically distributed. Thus annualized means and variances are 4 times the quarterly quantities, and annualized standard deviations are twice their quarterly counterparts.

Summary Statistics

	M	A	B	S_1	S_2	S_3	S_4
	VMMXX	VBMFX	VFINX	SOLNG	SOLNG	SOLNG	SOLNG
average return*	4.80	6.93	9.80	15.05	13.71	13.43	10.49
average reward*	0.00	2.13	5.00	10.25	8.91	8.63	5.69
stdev of reward*	0.00	4.12	17.84	9.54	9.56	9.67	11.49
Sharpe ratio*	undef	0.52	0.28	1.07	0.93	0.89	0.50
accumated return	152%	181%	204%	352%	313%	306%	234%

* = annualized

Looking at the above table, S_1 is the clear winner of the four SOLNG strategies. It had the highest return (either average or accumulated), the least risk (standard deviation

of reward), and the largest Sharpe ratio. (The Sharpe ratio here is the *ex post* ratio for the entire period: the average reward divided by the standard deviation of reward.) More significantly, S_1 returned substantially more than 500 Index at almost half the risk. By applying the S_1 SOLNG strategy over the past nine-or-so years, one could truly have beaten the market and slept well at night.*

7. Computation and generalization

Table 2 shows computed statistics for $A = \text{VBMFX}$ and $B = \text{VFINX}$ under the $S_1 = 4 \times \frac{1}{7} + 4 \times \frac{1}{14} + 4 \times \frac{1}{28}$ weight system. We have developed a “Two Funds Optimization” package (Norton [2002a]) to do these computations. After quarterly return data for a money market fund and two risky funds have been entered in an Excel worksheet, the user need simply push a button and the computations are done. In particular, if we were following the S_1 SOLNG strategy with the funds VMMXX , VBMFX , and VFINX , we would have reallocated 90% of our money to Total Bond Market and 10% to 500 Index at the beginning of the current quarter (2002q4). This wouldn’t have been much of a change from the previous 92–8% bond–stock allocation.

The ideas of this paper have been generalized to handle more than two risky funds in Norton [2002b]. Here we need:

Axiom 1. There is no riskless portfolio (of risky funds).

Axiom 2. The expected reward of a portfolio is a linear function of its risk.

(Axioms 1 and 2 are assured if the risks of the risky components are linearly independent.) Then, assuming some component fund has positive expected reward, the Sharpe-optimal long portfolio (the risk vector as opposed to the proportions of the components) is uniquely distinguished from other long portfolios by making minimum angle with the expected-reward gradient.

* This statement is tongue in cheek. We should at least add the typical mutual fund caveat: the fund’s (S_1 ’s) past performance does not necessarily indicate how the fund will perform in the future. Nevertheless, past performance does seem to show that

1. S_1 is a risk averse strategy.
2. S_1 is able to keep up with the (stock) market in good times (e.g., 1994–1997).
3. S_1 is able to hang onto its gains when the market turns south.

Table 1

Quarterly Total Returns

quarter	<i>M</i> VMMXX	<i>A</i> VBMFX	<i>B</i> VFINX	<i>S</i> ₁ SOLNG	<i>S</i> ₂ SOLNG	<i>S</i> ₃ SOLNG	<i>S</i> ₄ SOLNG
1994q1	1.27	-2.71	-3.84	-3.05	-2.99	-3.42	-3.84
1994q2	1.09	-1.01	0.40	-0.81	-0.79	1.09	1.09
1994q3	0.90	0.52	4.86	4.86	3.51	0.90	0.90
1994q4	0.76	0.55	-0.05	-0.05	-0.05	0.76	-0.05
1995q1	1.39	4.81	9.71	9.71	9.71	1.39	1.39
1995q2	1.42	6.01	9.49	9.49	9.49	9.49	9.49
1995q3	1.46	1.88	7.94	7.94	7.94	7.94	7.94
1995q4	1.43	4.41	6.01	6.01	6.01	6.01	6.01
1996q1	1.31	-1.91	5.34	5.34	5.34	5.20	5.34
1996q2	1.31	0.60	4.45	4.45	4.45	4.45	4.45
1996q3	1.27	1.79	3.05	3.05	3.05	3.05	3.05
1996q4	1.30	3.12	8.36	8.36	8.36	8.36	8.36
1997q1	1.36	-0.62	2.63	2.63	2.63	2.63	2.63
1997q2	1.35	3.56	17.41	17.41	17.41	17.41	17.41
1997q3	1.33	3.40	7.48	7.48	7.48	7.48	7.48
1997q4	1.29	2.83	2.84	2.84	2.84	2.84	2.84
1998q1	1.28	1.56	13.91	13.91	13.91	13.91	13.91
1998q2	1.34	2.36	3.29	2.98	3.29	2.45	3.29
1998q3	1.33	4.13	-9.95	-2.51	-9.95	2.47	-9.95
1998q4	1.32	0.31	21.39	3.10	6.58	2.34	0.31
1999q1	1.35	-0.43	4.98	0.45	1.25	0.14	4.98
1999q2	1.23	-0.98	7.00	0.39	0.86	0.23	7.00
1999q3	1.16	0.79	-6.25	-1.11	-1.27	-5.27	-6.25
1999q4	1.18	-0.13	14.96	14.96	4.30	14.96	1.18
2000q1	1.61	2.42	2.24	2.24	2.35	2.24	2.24
2000q2	1.61	1.48	-2.61	-0.38	0.52	-2.61	1.48
2000q3	1.52	3.07	-0.93	1.37	2.19	-0.93	1.52
2000q4	1.41	3.98	-7.81	1.11	1.68	2.38	3.98
2001q1	1.43	3.24	-11.90	1.68	0.83	3.24	3.24
2001q2	1.14	0.79	5.82	0.79	1.30	0.79	0.79
2001q3	0.90	4.29	-14.72	3.10	2.16	2.82	-14.72
2001q4	0.63	-0.08	10.65	0.73	0.99	0.73	-0.08
2002q1	0.46	0.06	0.24	0.08	0.06	0.08	0.24
2002q2	0.45	2.80	-13.43	1.21	2.05	0.50	0.45
2002q3	0.39	3.71	-17.22	1.93	2.46	1.44	3.71
avg return*	4.80	6.93	9.80	15.05	13.71	13.43	10.49
avg reward*	0.00	2.13	5.00	10.25	8.91	8.63	5.69
stdev of rwd*	0.00	4.12	17.84	9.54	9.56	9.67	11.49
Sharpe ratio*	undef	0.52	0.28	1.07	0.93	0.89	0.50
accum return	152%	181%	204%	352%	313%	306%	234%

* annualized

Table 2Computations (S_1 , quarterly*)

quarter	Expected rewards		Sharpe ratios		Correlations	Sharpe-optimal long proportions	
	\bar{r}_A	\bar{r}_B	s_A	s_B	$\text{cor}(A, B)$	p_O^A	p_O^B
1993q4	1.46	1.99	0.78	0.65	0.33	70%	30%
1994q1	0.68	0.65	0.27	0.22	0.72	86%	14%
1994q2	0.12	0.60	0.05	0.20	0.76	0%	100%
1994q3	-0.33	0.83	-0.14	0.26	0.68	0%	100%
1994q4	-0.37	0.22	-0.17	0.08	0.68	0%	100%
1995q1	0.35	1.77	0.15	0.50	0.71	0%	100%
1995q2	0.98	2.95	0.38	0.73	0.83	0%	100%
1995q3	0.88	3.45	0.35	0.82	0.81	0%	100%
1995q4	1.38	3.97	0.55	0.99	0.81	0%	100%
1996q1	0.70	4.00	0.26	1.16	0.62	0%	100%
1996q2	0.27	3.91	0.11	1.28	0.60	0%	100%
1996q3	0.26	3.50	0.11	1.15	0.62	0%	100%
1996q4	0.35	4.15	0.15	1.33	0.66	0%	100%
1997q1	0.32	3.93	0.15	1.47	0.70	0%	100%
1997q2	0.59	5.74	0.30	1.18	0.58	0%	100%
1997q3	0.85	6.12	0.41	1.30	0.57	0%	100%
1997q4	0.84	5.70	0.42	1.19	0.51	0%	100%
1998q1	1.02	6.97	0.63	1.35	0.33	34%	66%
1998q2	0.87	5.70	0.62	1.21	0.21	53%	47%
1998q3	1.08	3.36	0.71	0.44	-0.25	87%	13%
1998q4	0.66	5.70	0.41	0.60	-0.45	84%	16%
1999q1	0.57	5.13	0.35	0.55	-0.45	83%	17%
1999q2	0.13	5.12	0.07	0.57	-0.46	73%	27%
1999q3	-0.22	4.59	-0.13	0.50	-0.26	0%	100%
1999q4	-0.45	4.82	-0.29	0.54	-0.31	0%	100%
2000q1	-0.15	4.15	-0.10	0.47	-0.40	54%	46%
2000q2	-0.12	2.49	-0.09	0.28	-0.50	58%	42%
2000q3	-0.05	2.85	-0.04	0.34	-0.57	76%	24%
2000q4	0.39	-0.22	0.26	-0.03	-0.66	90%	10%
2001q1	0.65	-2.75	0.42	-0.32	-0.73	100%	0%
2001q2	0.65	-2.05	0.44	-0.23	-0.77	94%	6%
2001q3	0.93	-3.44	0.55	-0.35	-0.85	92%	8%
2001q4	0.73	-2.56	0.44	-0.27	-0.87	91%	9%
2002q1	0.58	-1.79	0.36	-0.20	-0.87	90%	10%
2002q2	1.02	-4.17	0.65	-0.44	-0.90	92%	8%
2002q3	1.21	-5.21	0.74	-0.51	-0.94	90%	10%

* to annualize, multiply expected rewards by 4 and Sharpe ratios by 2

References

- NORTON, JR., VICTOR T., Two Funds Optimization,
<<http://vic.norton.name/finance/btm/>>, 2002a.
- NORTON, JR., VICTOR T., The Geometry of Risk and Reward,
<<http://vic.norton.name/finance/grr/>>, 2002b.
- SHARPE, WILLIAM F., The Sharpe Ratio, *Journal of Portfolio Management*, **21** (Fall 1994), 49–58.
(also <<http://www.stanford.edu/~wfsharpe/art/sr/sr.htm>>)